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LONGEST CYCLES IN POLYHEDRAL GRAPHS

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ABSTRACT

A sequence of polyhedral graphs $\mathbf{G}_n$ is constructed, having only 3-valent and 8-valent vertices and having only 3-gons and 8-gons as faces with the property that the shortness exponent of the sequence as well as the shortness exponent of the sequence of duals is smaller than one.

We consider **polyhedral graphs**, that is graphs which are planar and 3-connected. For a graph $\mathbf{G} = \mathbf{G}(V, E)$ let $v(\mathbf{G})$, $f(\mathbf{G})$ and $c(\mathbf{G})$ be the number of vertices, the number of elementary faces and the **circumference** (the number of vertices of a longest cycle) of $\mathbf{G}$, resp. A graph $\mathbf{G}$ is called **hamiltonian** if $c(\mathbf{G}) = v(\mathbf{G})$. The **valency** $v(X)$ of a vertex $X \in V(\mathbf{G})$ is the number of edges incident to X. The **length** $l(F)$ of an elementary face F is the number of edges bordering F. A face F with $l(F) = i$ is called an i-gon. $v_i(\mathbf{G})$ and $f_i(\mathbf{G})$ are the number of vertices of $\mathbf{G}$ of valency i and the number of i-gons in $\mathbf{G}$, resp.

Let $\mathbf{\Gamma}$ be a family of graphs. The **shortness exponent** $\sigma(\mathbf{\Gamma})$ of $\mathbf{\Gamma}$ is defined [2] by

$$\sigma(\mathbf{\Gamma}) := \liminf_{\mathbf{G} \in \mathbf{\Gamma}} \frac{\log c(\mathbf{G})}{\log v(\mathbf{G})}.$$

Several families of polyhedral graphs with a shortness exponent smaller than one are known [2,3,4,5,6]. Moreover, let $\mathbf{\Gamma}^*$ be the family of polyhedral graphs $\mathbf{G}^*$ dual to $\mathbf{G} \in \mathbf{\Gamma}$.

Several families $\mathbf{\Gamma}$ are known with $\sigma(\mathbf{\Gamma}) < 1$ and $\sigma(\mathbf{\Gamma}^*) < 1$.

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In order to show $\sigma(\mathbf{\Gamma}) < 1$ there are generally constructed special sequences $\{\mathbf{G}_n\} \subset \mathbf{\Gamma}$ with the property

$$\lim_{n\to\infty} \frac{\log c(\mathbf{G}_n)}{\log v(\mathbf{G}_n)} < 1.$$

In case that $\sigma(\mathbf{\Gamma}^*) < 1$ holds, too, in general the sequence $\{\mathbf{G}_n^*\} \subset \mathbf{\Gamma}^*$ with $\mathbf{G}_n^*$ dual to $\mathbf{G}_n$ **does not** fulfill the inequality

$$\lim_{n\to\infty} \frac{\log c(\mathbf{G}_n^*)}{\log v(\mathbf{G}_n^*)} < 1.$$

Moreover, the existence of this limit is generally not guaranteed.

In the following, we consider sequences $\{\mathbf{G}_n\}$ of polyhedral graphs for which

$$\lim_{n\to\infty} \frac{\log c(\mathbf{G}_n)}{\log v(\mathbf{G}_n)}$$

and the corresponding limit for the sequence of the duals $\mathbf{G}_n^*$ of $\mathbf{G}_n$ exist. We will denote these limits by $\sigma\{\mathbf{G}_n\}$ and $\sigma\{\mathbf{G}_n^*\}$, resp.

We denote by $\mathbf{\Gamma}(p_1, p_2, \ldots, p_r; q_1, q_2, \ldots, q_s)$ with $p_1 < p_2 < \cdots < p_r$ and $q_1 < q_2 < \cdots < q_s$ the family of polyhedral graphs $\mathbf{G}$ with the following property: for any vertex $X \in V(\mathbf{G})$ there is an integer $k \in \{1, 2, \ldots, r\}$ with $v(X) = p_k$ and for any elementary face F of $\mathbf{G}$ there is an integer $j \in \{1, 2, \ldots, s\}$ with $l(F) = q_j$, and vice versa: To any p_k there is a vertex $X \in V(\mathbf{G})$ with $v(X) = p_k$ and to any q_l there is a face F with $l(F) = q_l$.

Obviously, if $\mathbf{G} \in \mathbf{\Gamma}(p_1, \ldots, p_r; q_1, \ldots, q_s)$, then the dual $\mathbf{G}^*$ of $\mathbf{G}$ is in $\mathbf{\Gamma}(q_1, \ldots, q_s; p_1, \ldots, p_r)$. We only consider families $\mathbf{\Gamma}$ of graphs with restricted valencies and restricted lengths of the elementary faces.

Definition: A class $\mathbf{\Gamma} = \mathbf{\Gamma}(p_1, \ldots, p_r; q_1, \ldots, q_s)$ of polyhedral graphs is called **minishort** if there exists a sequence $\{\mathbf{G}_n\} \subset \mathbf{\Gamma}$ such that $\sigma\{\mathbf{G}_n\} < 1$ and $\sigma\{\mathbf{G}_n^*\} < 1$. ∎

Trivially, $\mathbf{\Gamma}$ is minishort iff $\mathbf{\Gamma}^*$ is minishort.

For $\mathbf{\Gamma} = \mathbf{\Gamma}(p_1, p_2, \ldots, p_r; q_1, q_2, \ldots, q_s)$ let us shorten

$$b(\mathbf{\Gamma}) := \left| \{p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s\} \right|$$

and

$$d(\mathbf{\Gamma}) := r + s.$$

Obviously, $b(\mathbf{\Gamma}) = b(\mathbf{\Gamma}^*)$ and $d(\mathbf{\Gamma}) = d(\mathbf{\Gamma}^*)$.

PROBLEM: *What is the smallest integer b such that there exists a minishort family $\mathbf{\Gamma}$ with $b(\mathbf{\Gamma}) = b$?*

It is easy to see that $b \geq 2$ because there exists only one family $\mathbf{\Gamma}$ of polyhedral graphs with $b(\mathbf{\Gamma}) = 1$ namely, $\mathbf{\Gamma}(3;3)$ consisting of exactly one graph—the (*Platonic solid*) **tetrahedron**.

In case of $b(\mathbf{\Gamma}) = 2$ we have to distinguish three cases:

1. $d(\mathbf{\Gamma}) = 2$
   $\mathbf{\Gamma} = \mathbf{\Gamma}(p;q), \quad p \neq q.$
   There exist exactly 4 families $\mathbf{\Gamma}$ ($p = 3$, $q \in \{4,5\}$; $p \in \{4,5\}$, $q = 3$); each of them consists of exactly one element, namely one of the remaining Platonic solids. Each Platonic solid is hamiltonian.

2. $d(\mathbf{\Gamma}) = 3$

   - $\mathbf{\Gamma} = \mathbf{\Gamma}(p,q;p), \Longrightarrow p = 3$. In 1991 M. Tkáč [6] has shown that

     $$\liminf_{\mathbf{G}\in\mathbf{\Gamma}(3;3,q)} \frac{c(\mathbf{G})}{v(\mathbf{G})} < 1 \quad (7 \leq q \leq 10).$$

     For $q > 10$ it is easy to see that $\mathbf{\Gamma}(3;3,q)$ is empty.
     If $q < 7$ the shortness exponent equals 1 [1].

   - $\mathbf{\Gamma} = \mathbf{\Gamma}(p,q;q), \Longrightarrow p = 3$, $q \in \{4,5\}$. In 1972 G.Ewald [1] has shown that $\sigma(\mathbf{\Gamma}(3,4;3,4)) = 1$, that means $\mathbf{\Gamma}(3,4;4)$ is not minishort.
     In case of $\mathbf{\Gamma}(3,5;5)$ we have no information about minishortness.

3. $d(\mathbf{\Gamma}) = 4$
   For the family $\mathbf{\Gamma} = \mathbf{\Gamma}(p,q;p,q)$, $p = 3$ holds.

We can prove the following, which shows that the minimum b as defined above is two.

THEOREM: *The family $\mathbf{\Gamma} = \mathbf{\Gamma}(3,8;3,8)$ is minishort.*

Proof of the Theorem: We construct a suitable sequence $\{\mathbf{G}_n\}$ in the following way:

$\mathbf{G}_1$ is the polyhedral graph of Fig. 1. It is non hamiltonian because each of the 8 (white) vertices of valency 3 has only neighbours of valency 8(black vertices) and there are only 6 black vertices.

$\mathbf{G}_1 = \mathbf{G}_1(3,8;3,8)$

$v_3 = 8, v_8 = 6$

$f_3 = 24, f_8 = 0$

$c_3 = 6, c_8 = 6$

$c = 12$

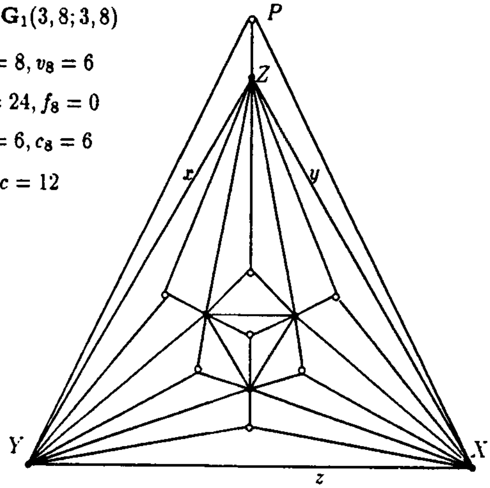

Figure 1

$\mathbf{G}_1^* = \mathbf{G}_1^*(3,8;3,8)$

$v_3 = 24, v_8 = 0$

$f_3 = 8, f_8 = 6$

$c_3 = 24, c_8 = 0$

$c = 24$

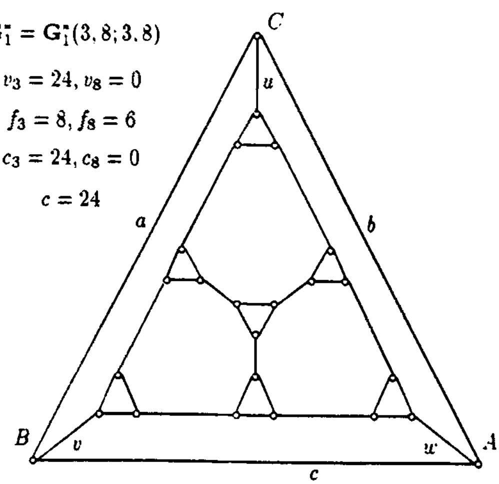

Figure 2

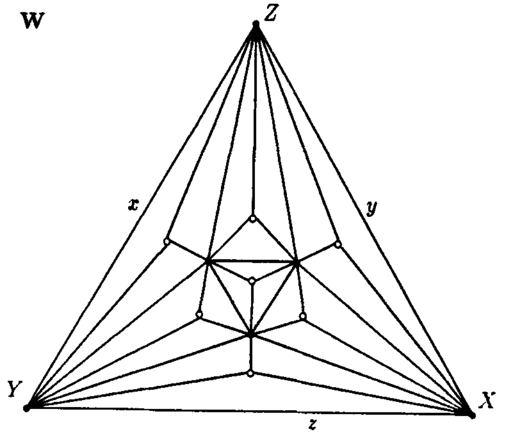

Figure 3

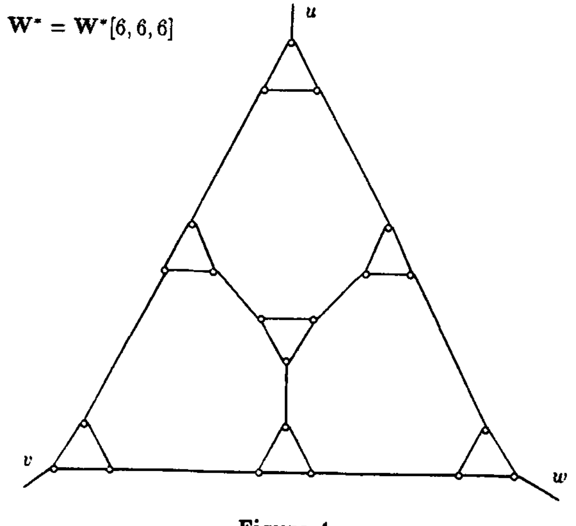

Figure 4

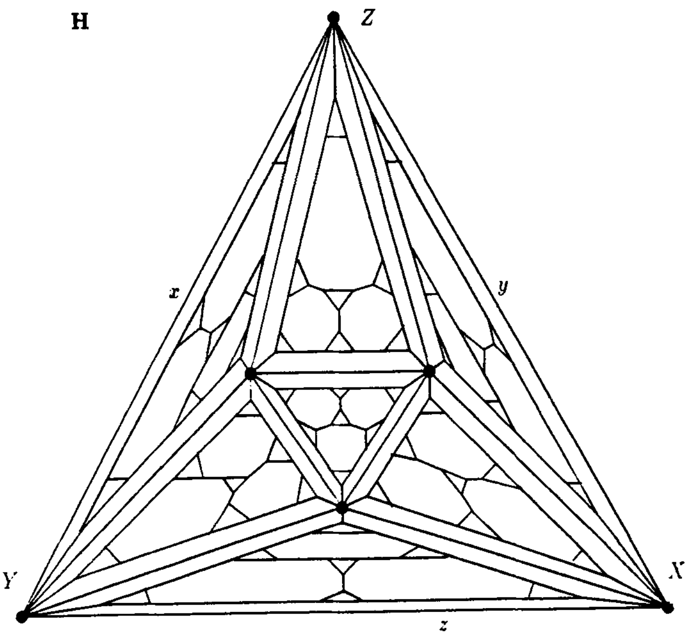

Figure 5

The dual $\mathbf{G}_1^*$ of $\mathbf{G}_1$ is drawn in Fig. 2. Cutting off the vertex P of $\mathbf{G}_1$ we obtain the graph $\mathbf{W}$ (see Fig. 3) and its **quasidual** $\mathbf{W}^*$(see Fig. 4. We obtain the real dual of $\mathbf{W}$ by inserting a further vertex incident with the three halfedges u, v, w). $\mathbf{W}^*[6,6,6]$ means that the number of vertices *between* any two of the three halfedges u, v, w along the border of $\mathbf{W}^*$ equals 6,6,6, resp.

If we replace in $\mathbf{W}$ each white vertex by a copy of $\mathbf{W}^*$, we obtain the graph $\mathbf{H}$ of Fig. 5. The quasidual graph $\mathbf{H}^*$ of $\mathbf{H}$ can be constructed by replacing each elementary triangle Δ of $\mathbf{W}^*$ by a copy of $\mathbf{W}$ identifying the three edges of Δ with the three edges x, y, z (see Fig. 6).

With the exception of the three vertices X, Y, Z in $\mathbf{W}$ and $\mathbf{H}$ all graphs and quasigraphs constructed up to now have only 3-valent and 8-valent vertices, and the length of any finite elementary face is 3 or 8.

$\mathbf{G}_{m+1}$ arises by replacing each vertex of valency 3 in $\mathbf{G}_m$ by a copy of $\mathbf{H}^*$.

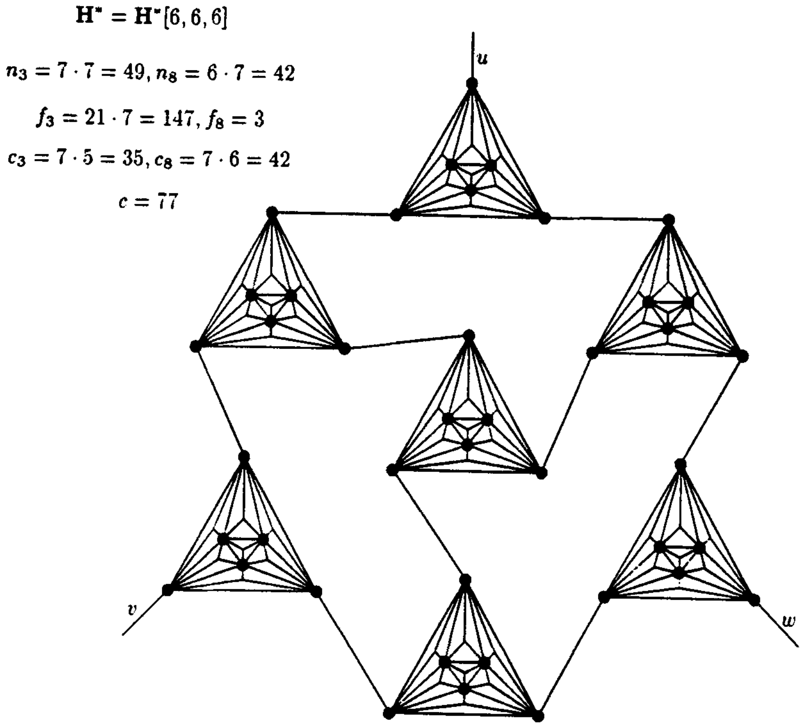

Figure 6

It is not difficult to see that $\mathbf{G}^*_{m+1}$ arises by replacing each elementary triangle Δ of $\mathbf{G}^*_m$ by a copy of $\mathbf{H}$ identifying the three edges of Δ with the three edges x, y, z of $\mathbf{H}$.

Now, we have to count the numbers $v(\mathbf{G}_m)$ and $v(\mathbf{G}^*_m)$ of vertices of $\mathbf{G}_m$ and $\mathbf{G}^*_m$, resp., as well as the numbers $c(\mathbf{G}_m)$ and $c(\mathbf{G}^*_m)$ of vertices contained in a longest cycle of $\mathbf{G}_m$ and $\mathbf{G}^*_m$, resp. Let $c_i(\mathbf{G})$ be the maximum number of i-valent vertices contained in a longest cycle of $\mathbf{G}$.

As shown in Fig. 1, we have

$v_3(\mathbf{G}_1) = 8,\ v_8(\mathbf{G}_1) = 6,\ f_3(\mathbf{G}_1) = 24,$

$c_3(\mathbf{G}_1) = c_8(\mathbf{G}_1) = 6,\ c(\mathbf{G}_1) = c_3(\mathbf{G}_1) + c_8(\mathbf{G}_1) = 12.$

In accordance with the construction given above, we get

$$v_3(\mathbf{G}_{m+1}) = 49 \cdot v_3(\mathbf{G}_m) = \cdots = 8 \cdot 49^m,$$
$$v_8(\mathbf{G}_{m+1}) = v_8(\mathbf{G}_m) + 42 \cdot v_3(\mathbf{G}_m) = \cdots = 7 \cdot 49^m - 1,$$
$$f_3(\mathbf{G}_{m+1}) = 3 \cdot 49 \cdot v_3(\mathbf{G}_m) = 24 \cdot 49^m,$$
$$f_8(\mathbf{G}_{m+1}) = f_8(\mathbf{G}_m) + 6 \cdot v_3(\mathbf{G}_m) = \cdots = 49^m - 1,$$
$$c_3(\mathbf{G}_{m+1}) = 7 \cdot 5 \cdot c_3(\mathbf{G}_m) = \cdots = c_3(\mathbf{G}_1) \cdot 35^m = 6 \cdot 35^m$$

because a longest path through **W** (Fig. 3) connecting any two of the three marginal vertices X, Y, Z contains 5 of the 7 vertices of valency 3.

Moreover, a longest path through $\mathbf{H}^*$ connecting any two of the three halfedges u, v, w contains all the $6 \cdot 7$ (black) vertices of valency 8 and $7 \cdot 5$ vertices of valency 3, that means

$$c_8(\mathbf{G}_{m+1}) = c_8(\mathbf{G}_m) + 6 \cdot 7 \cdot c_3(\mathbf{G}_m) = c_8(\mathbf{G}_1) + 6 \cdot 42 \cdot \frac{35^m - 1}{35 - 1} < 8 \cdot 35^m$$

$$\Longrightarrow c(\mathbf{G}_{m+1}) = c_8(\mathbf{G}_{m+1}) + c_3(\mathbf{G}_{m+1}) < 14 \cdot 35^m.$$

We obtain

$$\sigma\{\mathbf{G}_n\} = \lim_{m\to\infty} \frac{\log c(\mathbf{G}_m)}{\log v(\mathbf{G}_m)} \le \cdots \le \frac{\log 35}{\log 49}.$$

What about the sequence $\{\mathbf{G}_n^*\}$ of duals of $\mathbf{G}_n$?

Let C_m^* be a longest cycle of $\mathbf{G}_m^*$ and let Δ^* be any elementary triangle in $\mathbf{G}_m^*$ with the property that all of its three vertices are contained in C_m^*. We can *blow up* C_m^* to a C_{m+1}^* of $\mathbf{G}_{m+1}^*$ in the following way:

All the 6 vertices of valency 8 of **H** inserted in Δ^* are contained in C_{m+1}^* and $5 \cdot 21$ of the $7 \cdot 21$ vertices of valency 3 are contained in C_{m+1}^* (each vertex of valency 3 is contained in exactly one elementary triangle) and, if an 8-valent vertex of $\mathbf{G}_m^*$ is contained in C_m^*, then it occurs in C_{m+1}^*, too. We obtain

$$c_3(\mathbf{G}_{m+1}^*) = \tfrac{1}{3} \cdot 5 \cdot 21 \cdot c_3(\mathbf{G}_m^*) = \cdots = 24 \cdot 35^m,$$

$$c_8(\mathbf{G}_{m+1}^*) = c_8(\mathbf{G}_m^*) + \tfrac{1}{3} \cdot 6 \cdot c_3(\mathbf{G}_m^*) = c_8(\mathbf{G}_m^*) + 2 \cdot 24 \cdot 35^{m-1} = \cdots = \tfrac{48}{34}(35^m - 1),$$

$$c(\mathbf{G}_{m+1}^*) = c_3(\mathbf{G}_{m+1}^*) + c_8(\mathbf{G}_{m+1}^*) < \cdots < 26 \cdot 35^m,$$

and finally

$$\sigma\{\mathbf{G}_n^*\} \le \lim_{m\to\infty} \frac{\log(26 \cdot 35^m)}{\log(23 \cdot 49^m)} = \frac{\log 35}{\log 49}.$$

This completes the proof of the Theorem. ∎

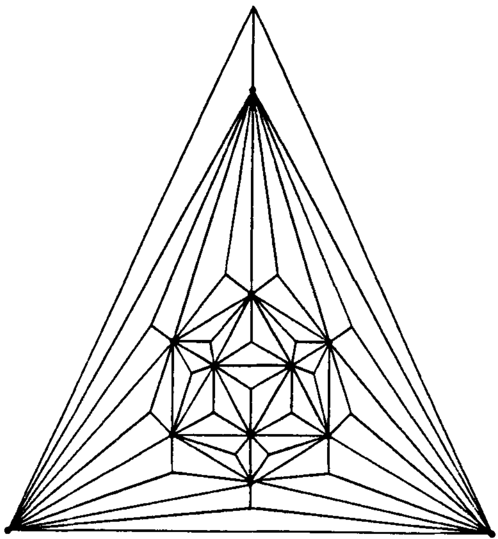

Figure 7

In an analogous way one can prove that $\Gamma(3,10;3,10)$ is minishort starting with the well-known non-hamiltonian graph of Fig. 7 and its dual.

We wish to raise the following conjectures and an open problem:

1. CONJECTURE: $\Gamma(3,5;5)$ *is not minishort.*

2. CONJECTURE: $\Gamma(3,q;3)$ *is not minishort.*

3. *What about the minishortness of the families*

$$\Gamma(3,q;3,q), \quad q = 7,9,11,12,\ldots ?$$

After submitting this paper P.J. Owens has constructed a sequence $\{\mathbf{P}_n\} \subset \Gamma(3,8;3,8)$ of selfdual polyhedral graphs with a shortness exponent smaller one (private communication of S. Jendrol).

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